...na 1.1

(i) All hyperplane are convex

(ii) Closed half spaces $\{\vec{x} \mid \vec{c}^T\vec{x} \le z\}$ or $\{\vec{x} \mid \vec{c}^T\vec{x} \ge z\}$ Convex

(iii) open half spaces $\{\vec{x} \mid \vec{c}^T\vec{x} < z\}$ or ______ convex

(iv) FS is convex

$x + y \le 50$

$40x + 60y \le 2400$

$$\boxed{A\vec{x} \begin{array}{l} \le \vec{b} \\ \ge \vec{b} \\ = \vec{b} \end{array}}$$

$$\begin{bmatrix} a_{11} & \vec{a}_1^T \cdots \vec{a}_2^T & a_{1n} \\ & & \\ a_{m1} & \vec{a}_m & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vec{x} \\ x_n \end{bmatrix} \begin{array}{c} \le \\ \ge \\ = \end{array} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

(iii)

Convex $\Leftarrow$ $\bigcap$ intersection

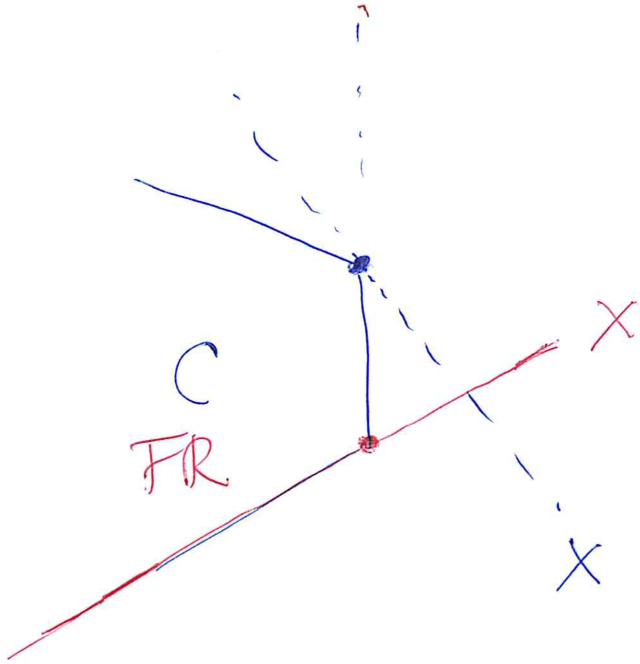
$\{\vec{x} : \vec{a}_1^T\vec{x} \ \ge \le = \ b_1\} \Rightarrow$ hyperplane (i), closed half space (ii), Convex

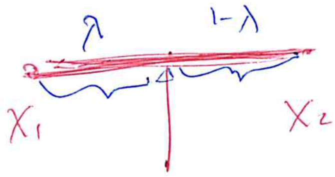
$\{\vec{x} : \vec{a}_2^T\vec{x} \ \ge \le = \ b_2\} \Rightarrow$

$\vdots$

$\{\vec{x} : \vec{a}_m^T\vec{x} \ \ge \le = \ b_m\} \Rightarrow$

$\underbrace{\qquad\qquad\qquad}_{FS}$ feasible region, feasible solution set





$$\vec{x} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2 \qquad \lambda \in [0,1]$$

$\lambda \geq 0,\ 1-\lambda \geq 0$

$\lambda + (1-\lambda) = 1$

"linear combination" of $\vec{x}_1, \dots, \vec{x}_n$

$$\vec{x} = \alpha_1 \vec{x}_1 + \dots + \alpha_n \vec{x}_n \qquad \alpha_i \in \mathbb{R}$$

$\vec{x}$ is "convex combination" of $\vec{x}_1, \dots, \vec{x}_n$

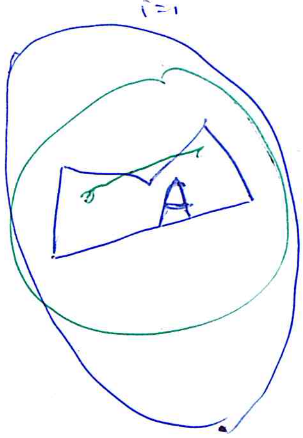
if $\vec{x} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \dots + \alpha_n \vec{x}_n$

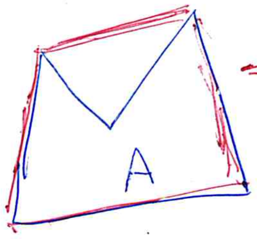
and $\alpha_1, \alpha_2 \dots \alpha_n \geq 0 \qquad \sum_{i=1}^{n} \alpha_i = 1$

convex hull of A

$$\langle A \rangle = \bigcap_{\substack{A \subseteq C \\ C \text{ convex}}} C$$

convex

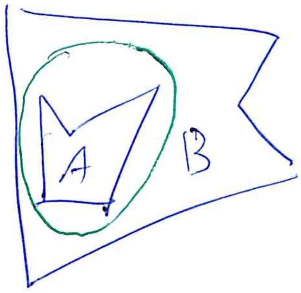




Lemma 1.1 (a) $A \neq \phi$, then $\langle A \rangle = \phi$

pf $\exists \vec{x} \in A \Rightarrow \vec{x} \in \langle A \rangle$ #

(b) $A \subseteq B$ then $\langle A \rangle \subseteq \langle B \rangle$



pf:

$$\langle A \rangle = \bigcap_{\substack{C \supseteq A \\ C \text{ convex}}} C = \bigcap_{\substack{C \supseteq B \\ \text{convex}}} C \cap \bigcap_{\substack{C \not\supseteq B \\ C \supseteq A \\ \text{convex}}} C$$

$$\subseteq \bigcap_{\substack{C \supseteq B \\ C \text{ convex}}} C = \langle B \rangle$$

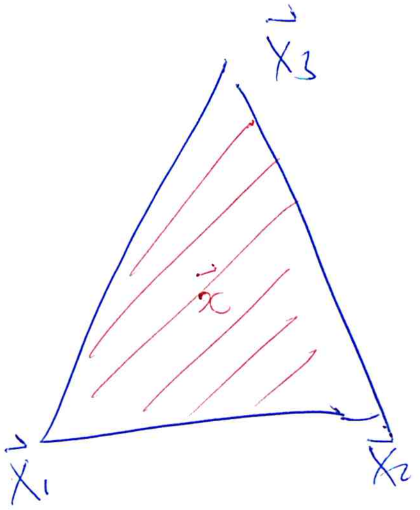
(c) $\langle A \rangle$ is the smallest convex set that contains A

pf: Contradiction If D smaller $\langle A \rangle$, D is convex $D \supseteq A$, then $D \cap \langle A \rangle$ smaller that $\langle A \rangle$.

(d) If A is convex, then $\langle A \rangle = A$.

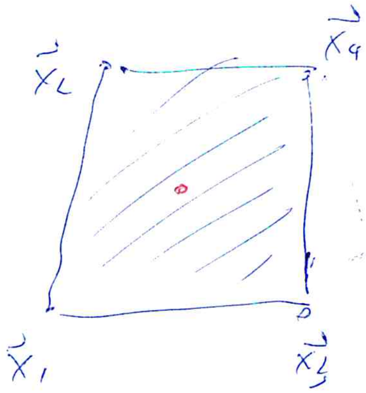
pf: By (c).

ie the convex hull is $D \cap \langle A \rangle$ rather than $\langle A \rangle$.



$\vec{x}$ c.c. of $\vec{X}_1, \vec{X}_2$ & $\vec{X}_3$.

$\langle \{\vec{X}_1, \vec{X}_2, \vec{X}_3\} \rangle$ = simplex = {set of all c.c. of $\vec{X}_1, \vec{X}_2$ & $\vec{X}_3$ }



Thm 1.2 $A = \{$ set of all c.c. of $\vec{X}_1, \ldots, \vec{X}_k \}$

$B = \langle \{X_1, \ldots, X_k\} \rangle$

$A = B$

(i) $B \subseteq A$ (ii) $A \subseteq B$ N.T.P.

(i) $B \subseteq A$:

$\boxed{\text{Stage 1: } A \text{ is convex}}$ $\left(\vec{y}_1, \vec{y}_2 \in A \Rightarrow \lambda \vec{y}_1 + (1-\lambda)\vec{y}_2 \in A \quad \forall \lambda \in [0,1] \right)$

$\vec{y}_1 = \sum_{i=1}^{k} M_i \vec{X}_i \qquad M_i \geq 0, \ \sum_{i=1}^{k} M_i = 1$

$\vec{y}_2 = \sum_{i=1}^{k} N_i \vec{X}_i \qquad N_i \geq 0 \quad \sum_{i=1}^{k} N_i = 1$

$\lambda \vec{y}_1 + (1-\lambda)\vec{y}_2 = \sum_{i=1}^{k} \underbrace{\{\lambda M_i + (1-\lambda) N_i\}}_{\geq 0} \vec{X}_i$ $\lambda \in [0,1]$

$\Uparrow$

A

$\sum_{i=1}^{k} (\lambda M_i + (1-\lambda) N_i) \overset{?}{=} 1$ o.k.

$\boxed{\text{Stage 2: } B \subseteq A}$ want to show $\{\vec{X}_1, \ldots, \vec{X}_k\} \subseteq A$

then $\langle \{\vec{X}_1, \ldots, \vec{X}_k\} \rangle \subseteq \langle A \rangle = A$ ($\because$ convex)

$|||$ Lemma 1.2 (b) Lemma 1.2 (d)

B

$\forall i \quad \vec{X}_i = 0\vec{X}_1 + 0\vec{X}_2 + \cdots + 0\vec{X}_{i-1} + 1\vec{X}_i + 0\vec{X}_{i+1} + \cdots + 0\vec{X}_k$

$\in A$

(ii) $A \subseteq B$ Mathematical Induction

Stage 1: $k=1$ is true

Stage 2 Assume $k-1$ is true $A_k \subseteq B_k = \langle\{\vec{x}_1, .. \vec{x}_k\}\rangle$

Stage 3 Prove that k is true $|||$

{set of co. co. of $\vec{x}_1, .. \vec{x}_k$}

Stage 1:

Suppose the set only has 1 pt $\{\vec{x}_1\}$

$$\lambda \vec{x}_1 + (1-\lambda)\vec{x}_1 = \vec{x}_1$$

$$A_1 = \{\vec{x}_1\} \quad B_1 = \langle\{\vec{x}_1\}\rangle = \{\vec{x}_1\} = A_1$$

#

Stage 2 Assume {set of any c.c. of $k-1$ pts} $\subseteq \langle\{k-1 \text{ points}\}\rangle$

Stage 3 $\underbrace{\{\text{set c.c. of } k \text{ pts } \vec{x}_1 \ldots, \vec{x}_k\}}_{= A_k} \underset{\text{N.T.P.}}{\subseteq} \langle\{\vec{x}_1, \ldots \vec{x}_k\}\rangle$

i.e. $\forall \vec{x} \in A_k \underset{\text{N.T.P.}}{\Longrightarrow} \vec{x} \in \langle\{\vec{x}_1 .. \vec{x}_k\}\rangle = B_k$??

Suppose $\vec{x} \in A_k$, $\vec{x} = \sum_{i=1}^{k} M_i \vec{x}_i \quad M_i \geq 0, \sum_{i=1}^{k} M_i = 1$ (0)

Case 1 $M_k = 1$ $\vec{x} = \vec{x}_k \in \langle\{\vec{x}_1, .. \vec{x}_k\}\rangle = B_k$

We are done!

Case 2 $M_k < 1 \Rightarrow 1 - M_k > 0$

define $\vec{y} = \sum_{i=1}^{k-1} \left(\frac{M_i}{1-M_k}\right) \vec{x}_i$ $\qquad \sum_{i=1}^{k-1}\left(\frac{M_i}{1-M_k}\right) = 1$

$\frac{M_i}{1-M_k} \geq 0$

$\Rightarrow \vec{y}$ is c.c. of $\{\vec{x}_1, \dots, \vec{x}_{k-1}\}$

$\Rightarrow \vec{y} \in A_{k-1} \subseteq B_{k-1} \subseteq B_k$ ①

induction hypothesis

$B_{k-1} = \langle\{\vec{x}_1, \dots \vec{x}_{k-1}\}\rangle \subseteq \langle\{\vec{x}_1, \dots \vec{x}_k\}\rangle = B_k$

Lemma 1.2(b)

$\because \{\vec{x}_1, \dots \vec{x}_{k-1}\} \subseteq \{\vec{x}_1, \dots, \vec{x}_{k-1}, \vec{x}_k\}$

On the other hand $\vec{x}_k \in B_k = \langle\{\vec{x}_1, \dots \vec{x}_k\}\rangle$ ②

Also B_k is convex ③

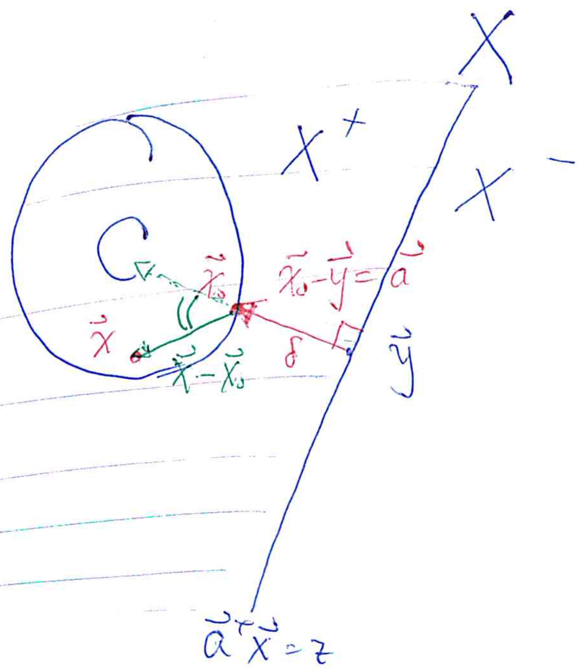
$\vec{x} \overset{⓪}{=} (1-M_k)\underset{B_k}{\underset{①\cap}{\vec{y}}} + M_k \underset{B_k}{\underset{②\cap}{\vec{x}_k}} \overset{③}{\in} B_k$ $\qquad M_k \in [0,1]$

$\therefore \vec{x} \in B_k$ #

Thm 1.3 If C closed & convex
$\vec{y} \notin C$

Then $\exists\, X = \{\vec{x} \mid \vec{a}^T\vec{x} = z\}$

s.t. $C \subseteq X^+ = \{\vec{x} \mid \vec{a}^T x \geq z\}$



Pf: $\delta = \min_{\vec{x} \in C} |\vec{x} - \vec{y}|$

min is attained $\because$ C is closed. $\quad \langle \vec{x} - \vec{x}_0, \vec{x}_0 - y \rangle$

minimum pt $\vec{x}_0 \in C$

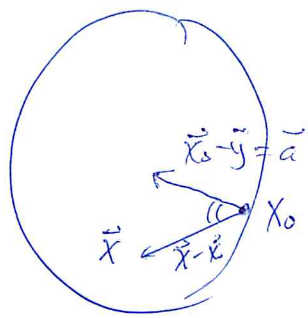
$\vec{a} \equiv \vec{x}_0 - \vec{y} \qquad \delta = |\vec{x}_0 - \vec{y}| \leq |\vec{x} - \vec{y}| \quad \forall \vec{x} \in C$ ①

$\vec{a}^T\vec{y} \equiv z$

$X \equiv \{\vec{x} \mid \vec{a}^T\vec{x} = \vec{a}^T\vec{y} = z\} \ni \vec{y}$

Claim $C \subseteq X^+ \equiv \{\vec{x} \mid \vec{a}^T\vec{x} > z\}$ N.T.P.

$\forall \vec{x} \in C \Rightarrow \vec{a}^T\vec{x} > z$ N.T.P



$\because$ C is convex, and $\vec{x}_0 \in C$, $\vec{x} \in C$

$\lambda\vec{x} + (1-\lambda)\vec{x}_0 \in C, \quad \forall \lambda \in [0,1]$

$\Rightarrow \underbrace{|\lambda\vec{x} + (1-\lambda)\vec{x}_0}_{C} - \vec{y}|^2 \overset{①}{\geq} |\vec{x}_0 - \vec{y}|^2 = \delta^2$

$\| \qquad \vec{a} \qquad\qquad \overset{\|}{a}$

$|\lambda(\vec{x} - \vec{x}_0) + (\vec{x}_0 - \vec{y})|^2$

pro

$$|\lambda(\vec{x}-\vec{x}_0)|^2 + 2\lambda\vec{a}^T(\vec{x}-\vec{x}_0) + \cancel{|a|^2} \geq \cancel{|a|^2} \qquad \lambda \in [0,1]$$

$$\Rightarrow \quad 2\lambda\vec{a}^T(\vec{x}-\vec{x}_0) + \lambda^2|\vec{x}-\vec{x}_0|^2 \geq 0 \quad \forall \lambda \in [0,1]$$

divide λ

$$\lambda \to 0 \quad \Rightarrow \quad 2\vec{a}^T(\vec{x}-\vec{x}_0) \geq 0$$

$$\begin{aligned}\vec{a}^T\vec{x} \geq \vec{a}^T\vec{x}_0 &= \vec{a}^T(\vec{a}+\vec{y})\\ &= \vec{a}^T\vec{a} + \vec{a}^T\vec{y}\\ &= |a|^2 + z = \delta^2 + z\\ &> z\end{aligned}$$

$$\therefore \quad \vec{a}^T\vec{x} > z. \quad \Rightarrow \quad \vec{x} \in X^+$$

\#

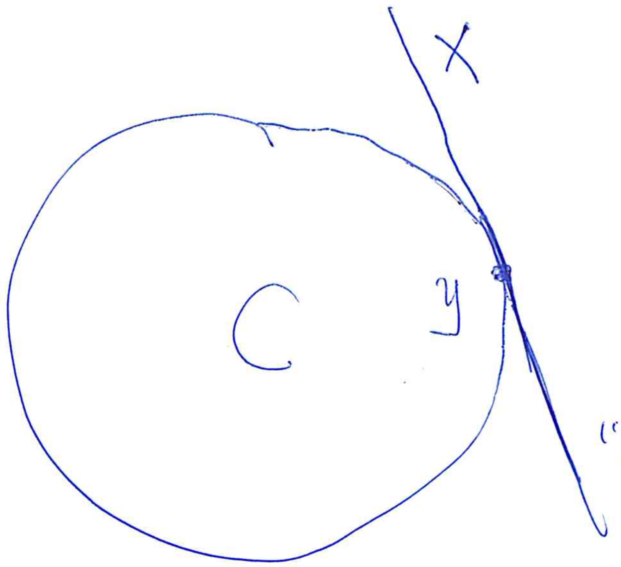
Thm If C closed & convex

$\vec{y}$ is a boundary pt of $C \Rightarrow \exists X$ st

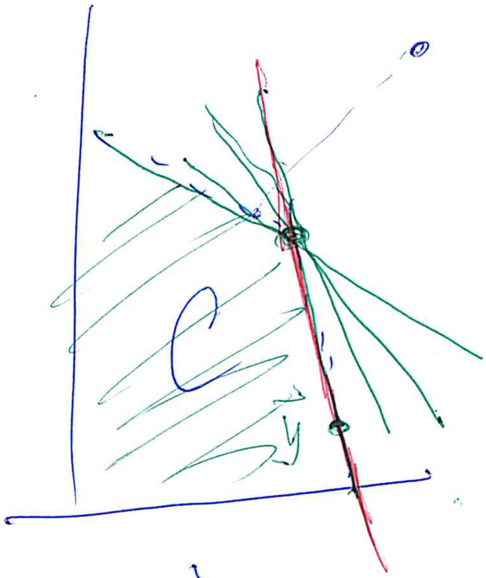
(i) $\vec{y} \in X = \{\vec{x} \mid \vec{c}^T\vec{x} = z\}$

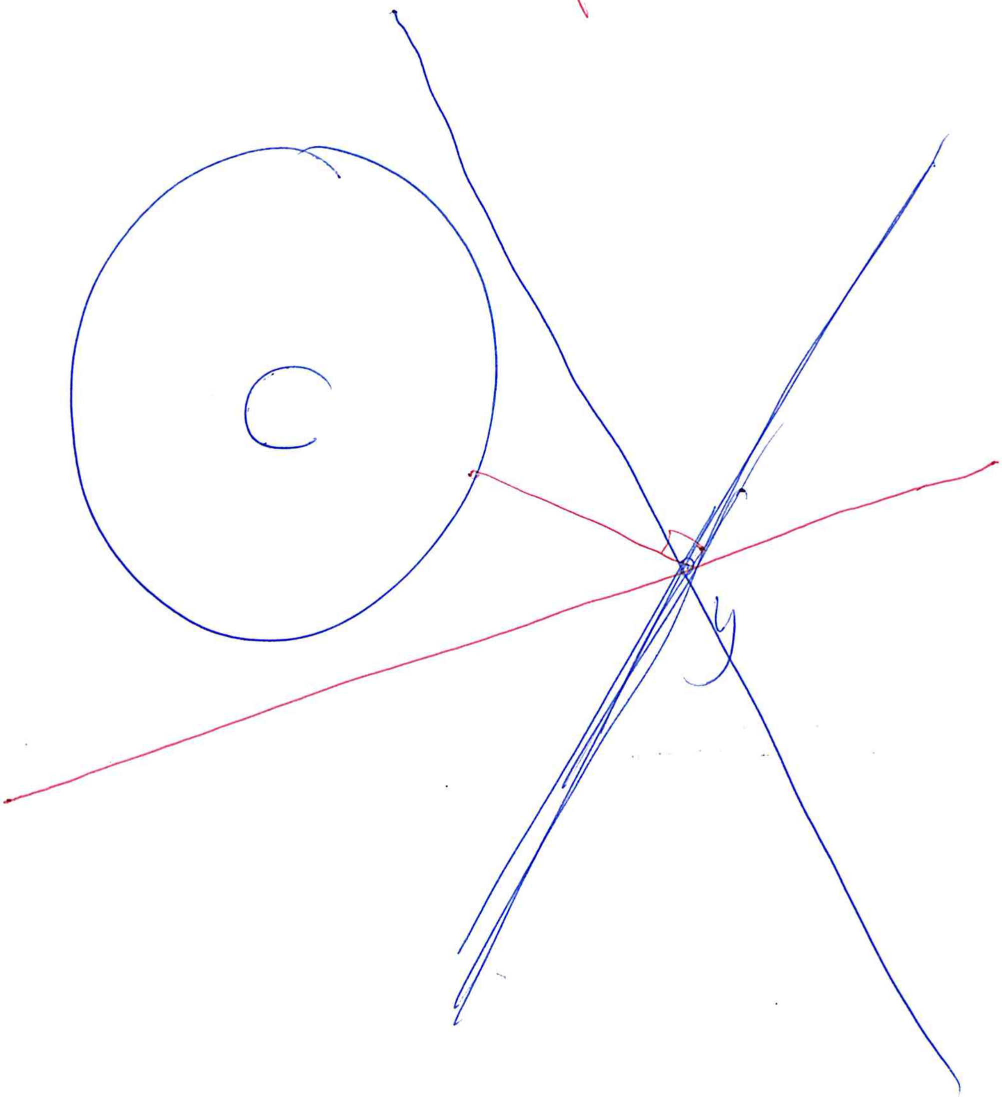
(ii) $C \subseteq X^+$

$= \{\vec{x} \mid \vec{c}^T\vec{x} \geq z\}$



"supporting hyperplane of C at $\vec{y}$"


C
y


C
y